DTT transform relationship derivation

Introduction.
In the paper **"The Algebraic Approach to the Discrete Cosine and Sine Transforms and their Fast Algorithms"** [1] an example of DTT relationship is described (see Example 8.4. DCT-1 and DST-2 on page 21). Though, some of details are not clear enough. Here I'm presenting original text (blue colour) of this example with my comments added (black colour).

Example 8.4. DCT-1 and DST-2.
We will translate a $\mathrm{DCT1}_{N+1}$ into a $\mathrm{DST2}_N$. Using again table 6.1 we get as associated algebras and bases

$$\mathrm{DCT1}_{N+1} \leftrightarrow C[x]\,/\,(x^2-1)U_{N-1}, \qquad b = (T_0, \dots, T_N)$$
$$\mathrm{DST2}_N \leftrightarrow C[x]\,/\,(x+1)U_{N-1}, \qquad b = (W_0, \dots, W_{N-1})$$

Note that we have to choose N+1 and N respectively, to obtain comparable algebras. We have $\mathrm{DCT1}_{N+1} = [T_l(\alpha_k)]_{k,l}$ and $\mathrm{DST2}_N = D \cdot [W_l(\alpha_k)]_{k,l}$, where $\alpha_k = cos\left(\frac{k\pi}{N}\right)$, $k = 0 \dots N$ are the zeros of $(x^2-1)U_{N-1}$ (2nd row in Table 4.2). For the $\mathrm{DST2}_N$, $\alpha_0 = 1$ is skipped.

This representation means that DCT1 matrix is defined in terms of Tschebyshev polynomials $\mathrm{T}_n(x)$, computed at roots of ring characteristic polynomial $\mathrm{P}_1(x) = (x^2-1)U_{N-1}$, and DST2 matrix is defined in terms of Tschebyshev polynomials $\mathrm{W}_n(x)$, computed at roots of ring characteristic polynomial $\mathrm{P}_2(x) = (x+1)U_{N-1}$.
According to the Table 4.2 roots are

$$\mathrm{P}_1(\alpha_k) = 0, \quad \alpha_k = cos\left(\frac{(\mathrm{k}+1)\pi}{N}\right), \quad k = -1 \dots N-1$$
$$\mathrm{P}_2(\alpha_k) = 0, \quad \alpha_k = cos\left(\frac{(\mathrm{k}+1)\pi}{N}\right), \quad k = 0 \dots N-1$$

Hence, matrix DCT1

$$\mathrm{DCT1}_{N+1} = [T_l(\alpha_k)]_{k,l} = \begin{bmatrix} T_0(\alpha_{-1}) & T_1(\alpha_{-1}) & \cdots & T_N(\alpha_{-1}) \\ T_0(\alpha_0) & T_1(\alpha_0) & & \\ \vdots & & \ddots & \\ T_0(\alpha_{N-1}) & & & T_N(\alpha_{N-1}) \end{bmatrix}$$

And matrix DST2

$$\mathrm{DST2}_N = D \cdot [W_l(\alpha_k)]_{k,l} = D \cdot \begin{bmatrix} W_0(\alpha_0) & W_1(\alpha_0) & \cdots & W_{N-1}(\alpha_0) \\ W_0(\alpha_1) & W_1(\alpha_1) & & \\ \vdots & & \ddots & \\ W_0(\alpha_{N-1}) & & & W_{N-1}(\alpha_{N-1}) \end{bmatrix}$$

Here for DCT1, k index runs from -1 to N, that is different from original text.

The scaling matrix is $\mathrm{D} = \mathrm{diag}_{k=0}^{N-1}\left(sin\left(\frac{(\mathrm{k}+1)\pi}{2N}\right)\right)$ (Table 6.2).

According to Table 6.1, scaling multiplier is $sin\left(\frac{1}{2}\theta\right)$. Value θ could be found from table 5.1 (see red lasso-line).

We compute the base change matrix B for

$$C[x]/(x^2-1)U_{N-1} \cong C[x]\,/\,(x-1) \oplus C[x]\,/\,(x+1)U_{N-1}$$

The bases are b, (1), b', respectively. Using $\mathrm{T}_l = \frac{1}{2}(\mathrm{W}_l - \mathrm{W}_{l-1})$ (4th row, 3rd column in Table 4.3) and $\mathrm{T}_N = \frac{1}{2}(\mathrm{W}_N - \mathrm{W}_{N-1}) \equiv -\mathrm{W}_{N-1} mod(x+1)U_{N-1}$ (because, again from Table 4.3, $(x+1)U_{N-1} = \frac{1}{2}(\mathrm{W}_N - \mathrm{W}_{N-1})$) we get

$$\mathrm{B} = \frac{1}{2} \cdot \begin{bmatrix} 2 & 2 & 2 & \cdot & \cdot & 2 \\ 2 & -1 & & & & \\ & 1 & -1 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & -1 & \\ & & & & 1 & -2 \end{bmatrix}$$

The 1's in the first row are due to $\mathrm{T}_l(1) = 1$ (Lemma 4.2). We get $[T_l(\alpha_k)]_{k,l} = \left(I_1 \oplus [W_l(\alpha_k)]_{k,l}\right) \cdot B$ and hence

$$(I_1 \oplus \mathrm{D}) \cdot \mathrm{DCT1}_{N+1} = (I_1 \oplus \mathrm{DST2}_N) \cdot B$$

Relation between $T_l(x)$ and $W_l(x)$ polynomials according to the table 4.3 is

$$\begin{bmatrix} T_0(x) \\ T_1(x) \\ \\ \\ T_{N-1}(x) \\ T_N(x) \end{bmatrix}^T = \begin{bmatrix} W_{-1}(x) \\ W_0(x) \\ W_1(x) \\ \\ \\ W_{N-1}(x) \\ W_N(x) \end{bmatrix}^T \cdot \frac{1}{2} \cdot \begin{bmatrix} -1 & & & & & \\ 1 & -1 & & & & \\ & 1 & -1 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & -1 \\ & & & & & 1 \end{bmatrix}$$

By green colour we marked basis elements out of boundaries of transform definition. Next step is finding symmetry relation for basis elements outside of boundaries.

According to the symmetry (table 4.2), $W_{-1}(x) = -W_0(x)$ and $W_N(x) = -W_{N-1}(x) mod(x+1)U_{N-1}$. So, this relation can be modified to

$$\begin{bmatrix} T_0(x) \\ T_1(x) \\ \\ \\ T_{N-1}(x) \\ T_N(x) \end{bmatrix}^T = \begin{bmatrix} W_0(x) \\ W_1(x) \\ \\ \\ W_{N-1}(x) \end{bmatrix}^T \cdot \frac{1}{2} \cdot \begin{bmatrix} 2 & -1 & & & & \\ & 1 & -1 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & -2 \end{bmatrix}$$

Substituting $\alpha_0 \ldots \alpha_{N-1}$ for x, we get relation between matrices generating DCT1 and DST2 transforms

$$\begin{bmatrix} T_0(\alpha_0) & T_1(\alpha_0) & & \\ \vdots & & \ddots & \\ T_0(\alpha_{N-1}) & & & T_N(\alpha_{N-1}) \end{bmatrix}$$

$$= \begin{bmatrix} W_0(\alpha_0) & W_1(\alpha_0) & \cdots & W_{N-1}(\alpha_0) \\ W_0(\alpha_1) & W_1(\alpha_1) & & \\ \vdots & & \ddots & \\ W_0(\alpha_{N-1}) & & & W_{N-1}(\alpha_{N-1}) \end{bmatrix} \cdot \frac{1}{2} \cdot \begin{bmatrix} 2 & -1 & & & & \\ & 1 & -1 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & -2 \end{bmatrix}$$

Taking into account that $T_l(\alpha_{-1}) = T_l(1) = 1$, we can add first row to the T-matrix in a next way

$$
\begin{bmatrix} T_0(\alpha_{-1}) & T_1(\alpha_{-1}) & \cdots & T_N(\alpha_{-1}) \\ T_0(\alpha_0) & T_1(\alpha_0) & & \\ \vdots & & \ddots & \\ T_0(\alpha_{N-1}) & & & T_N(\alpha_{N-1}) \end{bmatrix}
$$

$$
= \begin{bmatrix} 1 & \\ & \begin{bmatrix} W_0(\alpha_0) & W_1(\alpha_0) & \cdots & W_{N-1}(\alpha_0) \\ W_0(\alpha_1) & W_1(\alpha_1) & & \\ \vdots & & \ddots & \\ W_0(\alpha_{N-1}) & & & W_{N-1}(\alpha_{N-1}) \end{bmatrix} \end{bmatrix} \cdot \frac{1}{2} \cdot \begin{bmatrix} 2 & 2 & & \cdot & \cdot & 2 \\ 2 & -1 & & & & \\ & 1 & -1 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & -2 \end{bmatrix}
$$

Or

$$
[T_l(\alpha_k)]_{k=-1\ldots N-1, l=0..N} = \left(I_1 \oplus [W_l(\alpha_k)]_{k=0\ldots N-1, l=0\ldots N-1}\right) \cdot B
$$

By multiplication both sides by $(I_1 \oplus \mathrm{D}) \cdot \mathrm{DCT1}_{N+1}$, we can get final result

$$
(I_1 \oplus \mathrm{D}) \cdot \mathrm{DCT1}_{N+1} = (I_1 \oplus \mathrm{DST2}_N) \cdot B
$$

Appendix (Extracted tables)

	DCTs	DSTs
type 1	$\cos k\ell \frac{\pi}{n-1}$	$\sin(k+1)(\ell+1)\frac{\pi}{n+1}$
type 2	$\cos k(\ell+\frac{1}{2})\frac{\pi}{n}$	$\sin(k+1)(\ell+\frac{1}{2})\frac{\pi}{n}$
type 3	$\cos(k+\frac{1}{2})\ell\frac{\pi}{n}$	$\sin(k+\frac{1}{2})(\ell+1)\frac{\pi}{n}$
type 4	$\cos(k+\frac{1}{2})(\ell+\frac{1}{2})\frac{\pi}{n}$	$\sin(k+\frac{1}{2})(\ell+\frac{1}{2})\frac{\pi}{n}$
type 5	$\cos k\ell\frac{\pi}{n-\frac{1}{2}}$	$\sin(k+1)(\ell+1)\frac{\pi}{n+\frac{1}{2}}$
type 6	$\cos k(\ell+\frac{1}{2})\frac{\pi}{n-\frac{1}{2}}$	$\sin(k+1)(\ell+\frac{1}{2})\frac{\pi}{n+\frac{1}{2}}$
type 7	$\cos(k+\frac{1}{2})\ell\frac{\pi}{n-\frac{1}{2}}$	$\sin(k+\frac{1}{2})(\ell+1)\frac{\pi}{n+\frac{1}{2}}$
type 8	$\cos(k+\frac{1}{2})(\ell+\frac{1}{2})\frac{\pi}{n+\frac{1}{2}}$	$\sin(k+\frac{1}{2})(\ell+\frac{1}{2})\frac{\pi}{n-\frac{1}{2}}$

Table 5.1. (page 12) (θ is marked by red lasso-line)

	$P_n - P_{n-2}$	P_n	$P_n - P_{n-1}$	$P_n + P_{n-1}$
T_n	$2(x^2-1)U_{n-2}$	T_n	$(x-1)W_{n-1}$	$(x+1)V_{n-1}$
U_n	$2T_n$	U_n	V_n	W_n
V_n	$2(x-1)W_{n-1}$	V_n	$2(x-1)U_{n-1}$	$2T_n$
W_n	$2(x+1)V_{n-1}$	W_n	$2T_n$	$2(x+1)U_{n-1}$

Table 4.3 (page 11)

left b.c.	β_1	β_2
$a_{-1} = a_1$	0	2
$a_{-1} = 0$	0	1
$a_{-1} = a_0$	1	1
$a_{-1} = -a_0$	-1	1

right b.c.	β_3	β_4
$a_n = a_{n-2}$	2	0
$a_n = 0$	1	0
$a_n = a_{n-1}$	1	1
$a_n = -a_{n-1}$	1	-1

Table 5.3. (page 13)

			$a_n - a_{n-2}$	a_n	$a_n - a_{n-1}$	$a_n + a_{n-1}$
a_1	1	T_ℓ	DCT-1 $2(x^2-1)U_{n-2}$	DCT-3 T_n	DCT-5 $(x-1)W_{n-1}$	DCT-7 $(x+1)V_{n-1}$
0	$\sin\theta$	U_ℓ	DST-3 $2T_n$	DST-1 U_n	DST-7 V_n	DST-5 W_n
a_0	$\cos\frac{1}{2}\theta$	V_ℓ	DCT-6 $2(x-1)W_{n-1}$	DCT-8 V_n	DCT-2 $2(x-1)U_{n-1}$	DCT-4 $2T_n$
$-a_0$	$\sin\frac{1}{2}\theta$	W_ℓ	DST-8 $2(x+1)V_{n-1}$	DST-6 W_n	DST-4 $2T_n$	DST-2 $2(x+1)U_{n-1}$

Table 6.1 (page 16)

	$a_n = a_{n-2}$	$a_n = 0$	$a_n = a_{n-1}$	$a_n = -a_{n-1}$
$a_{-1} = a_1$	DCT-1	DCT-3	DCT-5	DCT-7
$a_{-1} = 0$	DST-3	DST-1	DST-7	DST-5
$a_{-1} = a_0$	DCT-6	DCT-8	DCT-2	DCT-4
$a_{-1} = -a_0$	DST-8	DST-6	DST-4	DST-2

Table 5.3. (page 13)

	$n=0,1$	closed form	symmetry	zeros	weight $w(x)$
T_n	$1, x$	$\cos(n\theta)$	$T_{-n}=T_n$	$\cos\frac{(k+\frac{1}{2})\pi}{n}$	$\frac{1}{(1-x^2)^{1/2}}$
U_n	$1, 2x$	$\frac{\sin(n+1)\theta}{\sin\theta}$	$U_{-n}=-U_{n-2}$	$\cos\frac{(k+1)\pi}{n+1}$	$(1-x^2)^{1/2}$
V_n	$1, 2x-1$	$\frac{\cos(n+\frac{1}{2})\theta}{\cos\frac{1}{2}\theta}$	$V_{-n}=V_{n-1}$	$\cos\frac{(k+\frac{1}{2})\pi}{n+\frac{1}{2}}$	$\frac{(1+x)^{1/2}}{(1-x)^{1/2}}$
W_n	$1, 2x+1$	$\frac{\sin(n+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}$	$W_{-n}=-W_{n-1}$	$\cos\frac{(k+1)\pi}{n+\frac{1}{2}}$	$\frac{(1-x)^{1/2}}{(1+x)^{1/2}}$

Table 4.2. (page 10)

Reference

1. **[The Algebraic Approach to the Discrete Cosine and Sine Transforms and their Fast Algorithms]**
SIAM Journal of Computing 2003, Vol. 32, No. 5, pp. 1280-1316